INVERSE OF A MATRIX

Definition

Let A be any square matrix. If there exists another square matrix B Such that AB = BA = I (I is a unit matrix) then B is called the inverse of the matrix A and is denoted by A^{-1}.

The cofactor method is used to find the inverse of a matrix. Using matrices, the solutions of simultaneous equations are found.

Working Rule to find the inverse of the matrix

Step 1: Find the determinant of the matrix.

Step 2: If the value of the determinant is non zero proceed to find the inverse of the matrix.

Step 3: Find the cofactor of each element and form the cofactor matrix.

Step 4: The transpose of the cofactor matrix is the adjoint matrix.

Step 5: The inverse of the matrix $A^{-1} = \dfrac{adj(A)}{|A|}$

Example

Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$

Solution

Let A = $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$

Step 1

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(9-3) + (4-2)$$

$$= 6 - 6 + 2 = 2 \neq 0$$

Step 2

The value of the determinant is non zero

$\therefore A^{-1}$ exists.

Step 3

Let A_{ij} denote the cofactor of a_{ij} in $|A|$

$$A_{11} = Cofactor\ of\ 1 = (-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 18 - 12 = 6$$

$$A_{12} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9-3) = -6$$

$$A_{13} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2$$

$$A_{21} = Cofactor\ of\ 1 = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9-4) = -5$$

$$A_{22} = Cofactor\ of\ 2 = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 9 - 1 = 8$$

$$A_{23} = Cofactor\ of\ 3 = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$$

$$A_{31} = Cofactor\ of\ 1 = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$$

$$A_{32} = Cofactor\ of\ 4 = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$A_{33} = Cofactor\ of\ 9 = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

Step 4

The matrix formed by cofactors of element of determinant $|A|$ is $\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$

$$\therefore \text{adj A} = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

Step 5

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & \frac{-5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & \frac{-3}{2} & \frac{1}{2} \end{pmatrix}$$

SOLUTION OF LINEAR EQUATIONS

Let us consider a system of linear equations with three unknowns

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

The matrix form of the equation is AX=B where $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a 3x3 matrix

X **=** $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B **=** $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Here AX = B

Pre multiplying both sides by A^{-1}.

$(A^{-1} A)X = A^{-1}B$

We know that $A^{-1} A = A A^{-1} = I$

$\therefore$ I X= $A^{-1}B$

since IX = X

Hence the solution is X = $A^{-1}B$.

Example

Solve the *x + y + z = 1,* 3*x* + 5*y* + 6*z = 4,* 9*x* + 26*y* + 36*z =16* by matrix method.

Solution

The given equations are *x + y + z = 1,*

3*x* + 5*y* + 6*z = 4,*

9*x* + 26*y* + 36*z =16*

Let A= $\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ 9 & 26 & 36 \end{pmatrix}$, X= $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, B= $\begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}$

The given system of equations can be put in the form of the matrix equation AX=B

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ 9 & 26 & 36 \end{vmatrix} = 1(180-156) - 1(108-54) + 1(78-45)$$

$$= 24 - 54 + 33 = 3 \neq 0$$

The value of the determinant is non zero

$\therefore A^{-1}$ exists.

Let A_{ij} (i, j = 1,2,3) denote the cofactor of a_{ij} in $|A|$

$$A_{11} = Cofactor\ of\ 1 = (-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 26 & 36 \end{vmatrix} = 180 - 156 = 24$$

$$A_{12} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix} 3 & 6 \\ 9 & 36 \end{vmatrix} = -(108 - 54) = -54$$

$$A_{13} = Cofactor\ of\ 1 = (-1)^{1+3}\begin{vmatrix} 3 & 5 \\ 9 & 26 \end{vmatrix} = 78 - 45 = 33$$

$$A_{21} = Cofactor\ of\ 3 = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 26 & 36 \end{vmatrix} = -(36 - 26) = -10$$

$$A_{22} = Cofactor\ of\ 5 = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 9 & 36 \end{vmatrix} = 36 - 9 = 27$$

$$A_{23} = Cofactor\ of\ 6 = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 9 & 26 \end{vmatrix} = -(26 - 9) = -17$$

$$A_{31} = Cofactor\ of\ 9 = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 5 & 6 \end{vmatrix} = 6 - 5 = 1$$

$$A_{32} = Cofactor\ of\ 26 = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} = -(6 - 3) = -3$$

$$A_{33} = Cofactor\ of\ 36 = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} = 5 - 3 = 2$$

The matrix formed by cofactors of element of determinant $|A|$ is $\begin{pmatrix} 24 & -54 & 33 \\ -10 & 27 & -17 \\ 1 & -3 & 2 \end{pmatrix}$

$$\therefore \text{adj } A = \begin{pmatrix} 24 & -10 & 1 \\ -54 & 27 & -3 \\ 33 & -17 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{3}\begin{pmatrix} 24 & -10 & 1 \\ -54 & 27 & -3 \\ 33 & -17 & 2 \end{pmatrix}$$

We Know that $X = A^{-1}B$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 24 & -10 & 1 \\ -54 & 27 & -3 \\ 33 & -17 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 24.1 + (-10)4 + 1.16 \\ (-54)1 + 27.4 + (-3)16 \\ 33.1 + (-17)4 + 2.16 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

***x* = 0, *y* = 2, *z* = -1.**

SOLUTION BY DETERMINANT (CRAMER'S RULE)

Let the equations be

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3, \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ (1)}$$

Consider the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

When $\Delta \neq 0$, the unique solution is given by

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}$$

Example

Solve the equations $x + 2y + 5z = 23$, $3x + y + 4z = 26$, $6x + y + 7z = 47$ by determinant method (Cramer's Rule).

Solution

The equations are

$x + 2y + 5z = 23,$

$3x + y + 4z = 26,$

$6x + y + 7z = 47$

$$\Delta = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 6 & 1 & 7 \end{vmatrix} = -6 \neq 0; \qquad \Delta_x = \begin{vmatrix} 23 & 2 & 5 \\ 26 & 1 & 4 \\ 47 & 1 & 7 \end{vmatrix} = -24;$$

$$\Delta_y = \begin{vmatrix} 1 & 23 & 5 \\ 3 & 26 & 4 \\ 6 & 47 & 7 \end{vmatrix} = -12 \qquad \Delta_z = \begin{vmatrix} 1 & 2 & 23 \\ 3 & 1 & 26 \\ 6 & 1 & 47 \end{vmatrix} = -18$$

By Cramer’s rule

$$x = \frac{\Delta_x}{\Delta} = \frac{-24}{-6} = 4$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-12}{-6} = 2$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-18}{-6} = 3$$

$\Rightarrow$ $x = 4, y = 2, z = 3.$